

Some open problems on cycles *

Chunhui Lai^{1,2}, *Mingjing Liu*¹

1. Department of Mathematics and Information Science,

Zhangzhou Normal University,

Zhangzhou, Fujian 363000, CHINA.

2. Center for Discrete Mathematics and Theoretical Computer Science,

Fuzhou University,

Fuzhou, Fujian 350003, CHINA.

e-mail: laich@winmail.cn; laichunhui@fjzs.edu.cn

(Chunhui Lai, Corresponding author)

liumingjing1983@163.com (Mingjing Liu)

Abstract

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. Erdős raised the problem of determining $f(n)$. Erdős conjectured that there exists a positive constant c such that $ex(n, C_{2k}) \geq cn^{1+1/k}$. Hajós conjecture that every simple even graph on n vertices can be decomposed into at most $n/2$ cycles. We present the problems, conjectures related to these problems and we summarize the know results. We do not think Hajós conjecture is true.

Key words: Hajós conjecture; even graph; Turan number; cycle; the maximum number of edges

AMS Subject Classifications: 05C35, 05C38

*Project supported by the National Science Foundation of China (No. 11101358), NSF of Fujian(2011J01026), Innovation Program for Young Scientists of Fujian(2011J05014), Fujian Provincial Training Foundation for "Bai-Quan-Wan Talents Engineering", IRTSTFJ, Project of Fujian Education Department (JA11165) and Project of Zhangzhou Teachers College.

Erdős Problem 1

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining $f(n)$ (see Bondy and Murty [1], p.247, Problem 11). Shi[41] proved a lower bound.

Theorem 1.1 (Shi[41])

$$f(n) \geq n + [(\sqrt{8n - 23} + 1)/2]$$

for $n \geq 3$.

Chen, Lehel, Jacobson, and Shreve [4], Jia[25], Lai[28,29,30,31,32,33,34], Shi[42,43,44,45,46,47], Y. Shi, Y. Tang, H. Tang, L. Gong and L. Xu[48] obtained some results.

Boros, Caro, Füredi and Yuster[3] proved an upper bound.

Theorem 1.2 (Boros, Caro, Füredi and Yuster[3]) For n sufficiently large,

$$f(n) < n + 1.98\sqrt{n}.$$

Lai [35] improved the lower bound.

Theorem 1.3 (Lai [35])

$$f(n) \geq n + \sqrt{2.4}\sqrt{n}(1 - o(1))$$

and proposed the following conjecture:

Conjecture 1.4 (Lai [35])

$$\lim_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} = \sqrt{2.4}.$$

It seems difficult to prove this conjecture. It would be nice to prove one of the following weaker conjectures:

Conjecture 1.5 (Lai[30])

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{3}.$$

Conjecture 1.6 (Lai[31])

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{2.4}.$$

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length.

Shi[44] proved that

Theorem 1.7 (Shi[44]) For every integer $n \geq 3$, $f_2(n) \leq n + [\frac{1}{2}(\sqrt{8n-15}-3)]$.

Chen, Lehel, Jacobson, and Shreve [4] proved that

Theorem 1.8 (Chen, Lehel, Jacobson, and Shreve [4]) $f_2(n) \geq n + \sqrt{n/2} - o(\sqrt{n})$

Boros, Caro, Füredi and Yuster [3] improved this lower bound significantly:

Theorem 1.9 (Boros, Caro, Füredi and Yuster [3]) $f_2(n) \geq n + \sqrt{n} - O(n^{\frac{9}{20}})$.

Corollary 1.10 (Boros, Caro, Füredi and Yuster [3])

$$\sqrt{2} \geq \limsup \frac{f_2(n) - n}{\sqrt{n}} \geq \liminf \frac{f_2(n) - n}{\sqrt{n}} \geq 1.$$

Boros, Caro, Füredi and Yuster [3] made the following conjecture:

Conjecture 1.11 (Boros, Caro, Füredi and Yuster [3])

$$\lim \frac{f_2(n) - n}{\sqrt{n}} = 1.$$

It is easy to see that Conjecture 1.11 implies the (difficult) upper bound in the Erdős-Turan Theorem [9,12](see Boros, Caro, Füredi and Yuster [3]).

Markström [27] raised the following problem:

Problem 1.12 (Markström [27]) Determine the maximum number of edges in a hamiltonian graph on n vertices with no repeated cycle lengths.

Let $g(n)$ denote the least number of edges of a graph which contains a cycle of length k for every $1 \leq k \leq n$. Jia[25] proved the following results:

Theorem 1.13 (Jia[25])

When n is sufficiently large,

$$n + \log_2 n - 1 \leq g(n) \leq n + \frac{3}{2} \log_2 n + 1.$$

Theorem 1.14 (Jia[25])

For a sufficiently large positive integer n , $g(n) \leq n + \log_2 n + \frac{3}{2} \log_2 \log_2 n + O(1)$

Corollary 1.15 (Jia[25])

For n sufficiently large, $g(n) = n + \log_2 n + O(\log_2 \log_2 n)$.

Jia[25] made the following conjecture:

Conjecture 1.16 (Jia[25])

$$g(n) = n + \log_2 n + O(1),$$

as $n \rightarrow \infty$.

The sequence (c_1, c_2, \dots, c_n) is the cycle length distribution of a graph G of order n where c_i is the number of cycles of length i in G . Let $f(a_1, a_2, \dots, a_n)$ denote the maximum possible number of edges in a graph which satisfies $c_i \leq a_i$ where a_i is a nonnegative integer. Shi posed the problem of determining $f(a_1, a_2, \dots, a_n)$ which extended the problem due to Erdős, it is clearly that $f(n) = f(1, 1, \dots, 1)$ (see Xu and Shi[52]).

The lower bound $f(0, 0, 2, \dots, 2)$ is given by Xu and Shi[52].

Theorem 1.17 (Xu and Shi[52]) For $n \geq 3$,

$$f(0, 0, 2, \dots, 2) \geq n - 1 + [(\sqrt{11n - 20})/2],$$

and the equality holds when $3 \leq n \leq 10$.

Given a graph H , what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turan number.

We denote by $m_i(n)$ the numbers of cycles of length i in the complete graph K_n on n vertices. Obviously,

$$\begin{aligned} ex(n, C_k) &= f(0, 0, m_3(n), \dots, \\ &\quad m_{k-1}(n), 0, m_{k+1}(n), \dots, m_n(n)) \\ &= f(0, 0, 2^{\frac{n(n-1)}{2}}, \dots, \\ &\quad 2^{\frac{n(n-1)}{2}}, 0, 2^{\frac{n(n-1)}{2}}, \dots, 2^{\frac{n(n-1)}{2}}). \end{aligned}$$

Therefore, finding $ex(n, C_k)$ is a special case of determining $f(a_1, a_2, \dots, a_n)$.

There are not good sufficient and necessary condition of when a graph on n vertices in which no two cycles have the same length. There are also not good sufficient and necessary condition of when a 2-connected graph on n vertices in which no two cycles have the same length.

Erdős conjecture 2

P. Erdős conjectured that there exists a positive constant c such that $ex(n, C_{2k}) \geq cn^{1+1/k}$ (see Erdős[11]). Erdős [8] posed the problem of determining $ex(n, C_4)$.

Erdős [10] (published without proof) and Bondy and Simonovits [2] obtained that

Theorem 2.1 (Erdős [10] and Bondy and Simonovits [2])

$$ex(n, C_{2k}) \leq ckn^{1+1/k}$$

Wenger [49] proved the following:

Theorem 2.2 (Wenger [49])

$$ex(n, C_4) \geq \left(\frac{n}{2}\right)^{3/2},$$

$$ex(n, C_6) \geq \left(\frac{n}{2}\right)^{4/3},$$

$$ex(n, C_{10}) \geq \left(\frac{n}{2}\right)^{6/5}.$$

Füredi[18] proved that

Theorem 2.3 (Füredi[18]) If q is a power of 2, then

$$ex(q^2 + q + 1, C_4) = q(q + 1)^2/2.$$

Füredi[19] also showed the following:

Theorem 2.4 (Füredi[19]) Let G be a quadrilateral-free graph with e edges on $q^2 + q + 1$ vertices, and suppose that $q \geq 15$. Then $e \leq q(q + 1)^2/2$.

Corollary 2.5 (Füredi[19]) If q is a prime power greater than 13, $n = q^2 + q + 1$. Then

$$ex(n, C_4) = q(q + 1)^2/2.$$

Füredi, Naor and Verstraete[20] proved that

Theorem 2.6 (Füredi, Naor and Verstraete[20])

$$ex(n, C_6) > 0.5338n^{4/3}$$

for infinitely many n and

$$ex(n, C_6) < 0.6272n^{4/3}$$

if n is sufficiently large.

This refuted the Erdős-Simonovits conjecture in 1982 for hexagons (see [20]).

The survey article on this Erdős conjecture can be found in Chung[5].

There are not good sufficient and necessary conditions of when a graph on n vertices contains k cycle. For $k = n$, it is Hamiltonian problem, the survey article on Hamiltonian problem can be found in Gould [21,22].

Hajós conjecture 3

An *eulerian graph* is a graph (not necessarily connected) in which each vertex has even degree. Let G be an eulerian graph. A *circuit decomposition* of G is a set of edge-disjoint circuits C_1, C_2, \dots, C_t such that $E(G) = C_1 \cup C_2 \cup \dots \cup C_t$. It is well known that every eulerian graph has a circuit decomposition. A natural question is to find the smallest number t such that G has a circuit decomposition of t circuits? Such smallest number t is called the *circuit decomposition number* of G , denoted by $cd(G)$. For each edge $xy \in E(G)$, let $m(xy)$ be the number of edges between x and y . The *multiple number* of G is defined by $m(G) = \sum_{uv \in E(G)} (m(uv) - 1)$. A

reduction of a graph G is a graph obtained from G by recursively applying the following operations:

1. Remove the edges of circuit.
2. Delete an isolated vertex (0-vertex).
3. Delete a 2-vertex with two distinct neighbors and add a new edge joining its two neighbors.
4. If u is a 4-vertex with 4 distinct neighbors $\{x, y, z, w\}$ such that $xy \in E(G)$ and $zw \notin E(G)$, then delete u and join x and y with a new parallel edge, and add a new edge between z and w .

A reduction is proper if it is not the original graph (see [16]). The following conjecture is due to Hajós (see [36]).

Hajós conjecture:

$$cd(G) \leq \frac{|V(G)|}{2}$$

for every simple eulerian graph G .

Lovasz [36] proved the following:

Theorem 3.1 (Lovasz [36]) A graph of n vertices can be covered by $\leq \lceil n/2 \rceil$ disjoint paths and circuits.

Jiang [26] and Seyffarth [40] considered planar eulerian graphs.

Theorem 3.2 (Jiang [26] and Seyffarth [40]) $cd(G) \leq \frac{|V(G)|-1}{2}$ for every simple planar eulerian graph G .

Granville and Moisiadis [23] and Favaron and Kouider [17] extended to multigraphs.

Theorem 3.3 (Granville and Moisiadis [23] and Favaron and Kouider [17]) If G is an even multigraph of order n , of size m , with $\Delta(G) \leq 4$, then $cd(G) \leq \frac{n+M-1}{2}$ where $M = m - m^*$ and m^* is the size of the simple graph induced by G .

Fan and Xu [16] proved that:

Theorem 3.4 (Fan and Xu [16]) If G is an eulerian graph with

$$cd(G) > \frac{|V(G)| + m(G) - 1}{2}$$

then G has a reduction H such that

$$cd(H) > \frac{|V(H)| + m(H) - 1}{2}$$

and the number of vertices of degree less than six in H plus $m(H)$ is at most one.

Corollary 3.5 (Fan and Xu [16]) Hajós conjecture is valid for projective graphs.

Corollary 3.6 (Fan and Xu [16]) Hajós conjecture is valid for K_6^- minor free graphs.

Xu[50] also proved the following two results:

Theorem 3.7 (Xu[50]) *If G is an eulerian graph with*

$$cd(G) > \frac{|V(G)| + m(G) - 1}{2}$$

such that

$$cd(H) \leq \frac{|V(H)| + m(H) - 1}{2}$$

for each proper reduction of G , then G is 3-connected. Moreover, if $S = \{x, y, z\}$ is a 3-cut of G , letting G_1 and G_2 be the two induced subgraph of G such that $V(G_1) \cap V(G_2) = S$ and $E(G_1) \cup E(G_2) = E(G)$, then either S is not an independent set, or G_1 and G_2 are both eulerian graphs.

Corollary 3.8 (Xu[50]) *To prove Hajós' conjecture, it suffices to prove*

$$cd(G) \leq \frac{|V(G)| + m(G) - 1}{2}$$

for every 3-connected eulerian graph G .

Fan [14] proved that

Theorem 3.9 (Fan [14]) Every eulerian graph on n vertices can be covered by at most $\lfloor \frac{n-1}{2} \rfloor$ circuits such that each edge is covered an odd number of times.

This settles a conjecture made by Chung in 1980(see[14]).

Xu and Wang[51] gave the following result:

Theorem 3.10 (Xu and Wang[51]) *The edge set of each even toroidal graph can be decomposed into at most $(n+3)/2$ circuits in $O(mn)$ time, where a toroidal graph is a graph embedable on the torus.*

Theorem 3.11 (Xu and Wang[51]) *The edge set of each toroidal graph can be decomposed into at most $3(n-1)/2$ circuits and edges in $O(mn)$ time.*

We do not think Hajós conjecture is true.

A related problem was conjectured by Gallai (see [36]):

Conjecture 3.12 (Gallai's conjecture) Every simple connected graph on n vertices can be decomposed into at most $(n+1)/2$ paths.

Lovasz [36] proved that

Theorem 3.13 (Lovasz [36]) If a graph has u odd vertices and g even vertices ($g \geq 1$), then it can be covered by $u/2 + g - 1$ disjoint paths.

Theorem 3.14 (Lovasz [36]) Let a locally finite graph have only odd vertices. Then it can be covered by disjoint finite paths so that every vertex is the endpoint of just one covering path.

The path number of a graph G , denoted $p(G)$, is the minimum number of edge-disjoint paths covering the edges of G . Donald [7] proved the following:

Theorem 3.15 (Donald [7]) If a graph with u vertices of odd degree and g nonisolated vertices of even degree. Then

$$p(G) \leq u/2 + \lceil \frac{3}{4}g \rceil \leq \lceil \frac{3}{4}n \rceil.$$

Pyber [37] proved that

Theorem 3.16 (Pyber [37]) A graph G of n vertices can be covered by $n - 1$ circuits and edges.

Theorem 3.17 (Pyber [37]) Let G be a graph of n vertices and $\{C_1, \dots, C_k\}$ be a set of circuits and edges such that $\cup_{i=1}^k E(C_i) = E(G)$ and k is minimal. Then we can choose k different edges, $e_i \in E(C_i)$, such that these edges form a forest in G .

Theorem 3.18 (Pyber [37]) Let G be a graph of n vertices not containing C_4 . Then G can be covered by $\lceil (n - 1)/2 \rceil$ circuits and $n - 1$ edges.

Pyber [38] proved that

Theorem 3.19 (Pyber [38]) Every connected graph G on n vertices can be covered by $n/2 + O(n^{3/4})$ paths.

Theorem 3.20 (Pyber [38]) Every connected graph on n vertices with e edges can be covered by $n/2 + 4(e/n)$ paths.

Reed [39] proved that

Theorem 3.21 (Reed [39]) Any connected cubic graph G of order n can be covered by $\lceil n/9 \rceil$ vertex disjoint paths.

Dean, Kouider [6] and Yan [53] proved that

Theorem 3.22 (Dean, Kouider [6] and Yan [53]) If a graph(possibly disconnected) with u vertices of odd degree and g nonisolated vertices of even degree. Then

$$p(G) \leq u/2 + \lceil \frac{2}{3}g \rceil.$$

Fan [13] proved that

Theorem 3.23 (Fan [13]) Every connected graph on n vertices can be covered by at most $\lceil n/2 \rceil$ paths.

This settles a conjecture made by Chung in 1980(see[13]).

Theorem 3.24 (Fan [13]) Every 2-connected graph on n vertices can be covered by at most $\lfloor \frac{2n-1}{3} \rfloor$ circuits.

This settles a conjecture made by Bondy in 1990(see[13]).

Corollary 3.25 (Fan [13]) Let G be a 2-edge-connected graph on n vertices. Then G can be covered by at most $\lfloor \frac{3(n-1)}{4} \rfloor$ circuits.

Fan [15] define a graph operation, called α -operation and proved that

Definition 3.26(Fan [15]) Let H be a graph. A pair (S, y) , consisting of an independent set S and a vertex $y \in S$, is called an α -pair if the following holds: for every vertex $v \in S \setminus \{y\}$, if $d_H(v) \geq 2$, then (a) $d_H(u) \leq 3$ for all $u \in N_H(v)$ and (b) $d_H(u) = 3$ for at most two vertices $u \in N_H(v)$. (That is, all the neighbors of v have degree at most 3, at most two of which have degree exactly 3.) An α -operation on H is either (i) add an isolated vertex or (ii) pick an α -pair (S, y) and add a vertex x joined to each vertex of S , in which case the ordered triple (x, S, y) is called the α -triple of the α -operation.

Definition 3.27(Fan [15]) An α -graph is a graph that can be obtained from the empty set via a sequence of α -operations.

Theorem 3.28 (Fan [15]) Let G be a graph on n vertices (not necessarily connected). The E-subgraph of G is the subgraph induced by the vertices of even degree in G . If the E-subgraph of G is an α -graph, then G can be decomposed into $\lfloor n/2 \rfloor$ paths.

Corollary 3.29 (Fan [15]) Let G be a graph on n vertices (not necessarily connected). If each block of the E-subgraph of G is a triangle-free graph with maximum degree at most 3, then G can be decomposed into $\lfloor n/2 \rfloor$ paths.

Harding and McGuinness [24] proved that

Theorem 3.30 (Harding and McGuinness [24]) For every simple graph G have girth $g \geq 4$, with u vertices of odd degree and w nonisolated vertices of even degree, there is a path-decomposition having at most $u/2 + \lfloor \frac{g+1}{2g} w \rfloor$ paths.

Acknowledgment

The authors would like to thank Professor Y. Caro, G. Fan, R. Gould, B. Xu, R. Yuster for their advice and sending some papers to us. The authors would like to thank the referee for his many valuable comments and suggestions.

References

- [1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976). MR0411988 (54 #117)
- [2] J.A Bondy and M. Simonovits, Cycle of even length in graphs, J. Combin. Theory Ser. B 16(1974),97-105. MR0340095 (49 #4851)
- [3] E. Boros, Y. Caro, Z. Füredi and R. Yuster, Covering non-uniform hypergraphs, J. Combin. Theory Ser. B 82(2001), 270-284. MR1842115 (2002g:05130)
- [4] G. Chen, J. Lehel, M. S. Jacobson and W. E. Shreve, Note on graphs without repeated cycle lengths, J. Graph Theory 29(1998),11-15. MR1633908 (2000d:05059)
- [5] F.R.K.Chung, Open problems of Paul Erdos in graph theory, J. Graph Theory 25 (1997), 3-36. MR1441976 (98e:05027)
- [6] N. Dean and M. Kouider, Gallai's conjecture for disconnected graphs, Selected topics in discrete mathematics (Warsaw, 1996). Discrete Math. 213 (2000), no. 1-3, 43-54. MR1755410 (2001a:05083)
- [7] A. Donald, An upper bound for the path number of a graph, J. Graph Theory 4 (1980), no. 2, 189-201. MR0570353 (81f:05102)
- [8] P. Erdos, On sequences of integers no one of which divides the product of two others and some related problems, Izv. Naustno-Issl. Mat. i Meh. Tomsk 2(1938), 74-82.
- [9] P. Erdos , On a problem of Sidon in additive number theory and on some related problems. Addendum, J. Lond. Math. Soc. 19 (1944), 208. MR0014111 (7,242f)
- [10] P. Erdos, Extremal problemes in graph theory, in Theory of graphs and its applications, Proc. Symp. (Smolenice, 1963), (M. Fiedler, ed.), 29-36. New York: Academic Press, 1965.

- [11] P. Erdos, Some unsolved problems in graph theory and combinatorial analysis, *Combinatorial Mathematics and its Applications* (Proc. Conf., Oxford, 1969) , pp. 97–109, Academic Press, London, 1971. MR0277392 (43 #3125)
- [12] P. Erdos and P. Turan, On a problem of Sidon in additive number theory, and on some related problems, *J. Lond. Math. Soc.* 16 (1941), 212–215. MR0006197 (3,270e)
- [13] G. Fan, Subgraph coverings and edge switchings, *J. Combin. Theory Ser. B* 84 (2002), no. 1, 54–83. MR1877901 (2002k:05187)
- [14] G. Fan, Covers of Eulerian graphs, *J. Combin. Theory Ser. B* 89 (2003), no. 2, 173 - 187. MR2017722 (2004m:05205)
- [15] G. Fan, Path decompositions and Gallai’s conjecture, *J. Combin. Theory Ser. B* 93 (2005), no. 2, 117–125. MR2117933 (2005m:05125)
- [16] G. Fan and B. Xu, Hajós conjecture and projective graphs, *Discrete Math.* 252 (2002), 91 - 101. MR1907748 (2003c:05175)
- [17] O. Favaron and M. Kouider, Path partitions and cycle partitions of eulerian graphs of maximum degree 4, *Studia Sci. Math. Hungar.* 23 (1988), 237 - 244. MR0962453 (89i:05213)
- [18] Z. Furedi, Graph without quadrilaterals, *J. Combin. Theory Ser. B* 34(1983), 187-190. MR0703603 (85b:05104)
- [19] Z. Furedi, On the number of edges of quadrilateral-free graphs, *J. Combin. Theory Ser. B* 68(1996), 1-6. MR1405701 (97g:05101)
- [20] Z. Furedi, A. Naor, J. Verstraete, on the Turan number for the hexagon, *Adv. Math.* 203(2) (2006), 476-496. MR2227729 (2007b:05104)
- [21] R. Gould, Updating the Hamiltonian problem - a survey, *J. Graph Theory* 15 (1991), no. 2, 121–157. MR1106528 (92m:05128)
- [22] R. Gould, Advances on the Hamiltonian problem - a survey, *Graphs Combin.* 19 (2003), no. 1, 7–52. MR1974368 (2004a:05092)
- [23] A. Granville and A. Moisiadis, On Hajós conjecture, in: *Proceedings of the 16th Manitoba Conference on Numerical Mathematics and Computing*, *Congr. Numer.* 56 (1987), 183 - 187. MR889695 (88d:05120)
- [24] P. Harding and S. McGuinness, Gallai’s conjecture for graphs of girth at least four, preprint

- [25] X. Jia, Some extremal problems on cycle distributed graphs, Congr. Numer. 121(1996), 216-222. MR1431994 (97i:05068)
- [26] T. Jiang, On Hajós conjecture, J. China Univ. Sci. Tech. 14 (1984), 585 - 592. MR0840783 (87g:05178)
- [27] K. Markström, A note on uniquely pancyclic graphs, Australas. J. Combin. 44 (2009), 105-110. MR2527003 (2010i:05180)
- [28] C. Lai, On the Erdős problem, J. Zhangzhou Teachers College(Natural Science) 3(1)(1989), 55-59. MR1025502 (91a:05063)
- [29] C. Lai, Upper bound and lower bound of $f(n)$, J. Zhangzhou Teachers College(Natural Science) 4(1)(1990), 30-34,29. MR1078010
- [30] C. Lai, On the size of graphs with all cycle having distinct length, Discrete Math. 122(1993), 363-364. MR1246693 (94i:05048)
- [31] C. Lai, On the maximum number of edges in a graph in which no two cycles have the same length, Combinatorics, graph theory, algorithms and applications (Beijing, 1993), 447-450, World Sci. Publ., River Edge, NJ, 1994. MR1313999 (96b:05080)
- [32] C. Lai, The edges in a graph in which no two cycles have the same length, J. Zhangzhou Teachers College (Natural Science) 8(4)(1994), 30-34. MR1336586 (96h:05121)
- [33] C. Lai, The numbers of edge of one class graphs, J. of Zhangzhou Teachers College (Natural Science) 12(2)(1999), 7-9, 20. MR1723698 (2000g:05080)
- [34] C. Lai, A lower bound for the number of edges in a graph containing no two cycles of the same length, Electron. J. Combin. 8(2001), #N9, 1 - 6. MR1877662 (2002k:05124)
- [35] C. Lai, Graphs without repeated cycle lengths, Australas. J. Combin. 27(2003), 101-105. MR1955391 (2003m:05102)
- [36] L. Lovasz, On covering of graphs, in: P. Erdos, G.O.H. Katona (Eds.), Theory of Graphs, Academic Press, New York, 1968, pp. 231 - 236. MR0233723 (38 #2044)
- [37] L. Pyber, An Erdos-Gallai conjecture, Combinatorica 5 (1985), no. 1, 67-79. MR0803241 (87a:05099)
- [38] L. Pyber, Covering the edges of a connected graph by paths, J. Combin. Theory Ser. B 66 (1996), no. 1, 152-159. MR1368522 (96i:05101)

- [39] B. Reed, Paths, stars and the number three, *Combin. Probab. Comput.* 5 (1996), no. 3, 277–295. MR1411088 (97k:05160)
- [40] K. Seyffarth, Hajós conjecture and small cycle double covers of planar graphs, *Discrete Math.* 101 (1992), 291 - 306. MR1172385 (93i:05104)
- [41] Y. Shi, On maximum cycle-distributed graphs, *Discrete Math.* 71(1988), 57-71. MR0954686 (89i:05169)
- [42] Y. Shi, The number of edges in a maximum graphs with uni-length cycles, *Kexue Tongbao* 33(10)(1988), 795 - 796.
- [43] Y. Shi, Some theorems on simple MCD-graphs, *J. Math. Res. Exposition* 11(4)(1991), 551-554. MR1144956 (92i:05182)
- [44] Y. Shi, The number of edges in a maximum cycle distributed graph, *Discrete Math.* 104(1992), 205-209. MR1172850 (93d:05083)
- [45] Y. Shi, On simple MCD-graphs, *Chinese Quarterly J. Math.* 7(3)(1992), 41-47.
- [46] Y. Shi, On simple MCD graphs containing a subgraph homomorphic to K_4 , *Discrete Math.* 126(1994), 325-338. MR1264498 (95f:05072)
- [47] Y. Shi, An upper bound on the size of simple MCD-graphs, *Pure Appl. Math.* 10(1994), Special Issue, 210-216. MR1369978 (96j:05063)
- [48] Y. Shi, Y. Tang, H. Tang, L. Gong and L. Xu, Two classes of simple MCD graphs, *Discrete geometry, combinatorics and graph theory*, 177-188, *Lecture Notes in Comput. Sci.*, 4381, Springer, Berlin, 2007. MR2364761 (2008j:05188)
- [49] R. Wenger, Extremal graphs with no C^4, s , C^6, s , or C^{10}, s , *J. Combin. Theory Ser. B* 52(1991), 113-116. MR1109426 (92c:05085)
- [50] B. Xu, Hajós conjecture and connectivity of Eulerian graphs, *J. Syst. Sci. Complex.* 15(3) (2002), 295 - 298. MR1925493 (2003c:05135)
- [51] B. Xu and L. Wang, Decomposing toroidal graphs into circuits and edges. *Discrete Appl. Math.* 148(2) (2005), 147 - 159. MR2134118 (2006a:05136)
- [52] J. Xu and Y. Shi, The maximum possible number of edges in a simple graph with at most two cycles having the same length, *J. Shanghai Normal Univ.(Natural Sciences)* 32(3)(2003), 26-32.
- [53] L. Yan, On path decompositions of graphs, Thesis (Ph.D.), Arizona State University. ProQuest LLC, Ann Arbor, MI, 1998. 42 pp. ISBN: 978-0591-80735-6 MR2697319